Quantum mechanics and elements of reality inferred from joint measurements

Quantum mechanics and elements of reality inferred from joint measurements

Adán Cabello and Guillermo García-Alcaine

Departamento de Física Teórica, Universidad Complutense, 28040 Madrid, Spain

Abstract. The Einstein–Podolsky–Rosen argument on quantum mechanics incompleteness is formulated in terms of elements of reality inferred from *joint* (as opposed to *alternative*) measurements, in two examples involving entangled states of three spin-$\frac{1}{2}$ particles. The same states allow us to obtain proofs of the incompatibility between quantum mechanics and elements of reality.

1. Introduction

The layout of the paper is as follows. In section 2 we briefly review some opinions on a controversial point in Einstein–Podolsky–Rosen's (EPR) [1] argument. In section 3 we introduce some notations and a more detailed form of the sufficient condition for joint inference of several elements of reality (ERs) in the same individual system. In section 4 we formulate EPR's incompleteness argument in terms of ERs inferred from joint measurements (instead of alternative incompatible measurements as in the original paper and all its sequels), in two entangled states of three spin-$\frac{1}{2}$ particles: an extension to three particles of Hardy's two-particle state [2], and the GHZ–Mermin state [3,4]. In section 5 we show the incompatibility between quantum mechanics (QM) and ERs (Bell's theorem), in the same entangled states used in section 4. Finally we summarize our conclusions in section 6.

2. Elements of reality and EPR's incompleteness argument

In 1935, EPR presented an argument to prove that QM only provides an 'incomplete' description of physical reality [1]. Their conclusions were striking because of 'the very mild character of the sufficient condition for the reality of a physical quantity on which their argument hinged' [5]: 'If, without in any way disturbing a system, we can predict with certainty (i.e. with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.'

There is a problem, however, in EPR's argument '...one cannot make both measurements [position or momentum of one of the particles in the original EPR example], and hence both predictions [position or momentum of the other particle], simultaneously' [6]. The joint ('simultaneous') existence in the same individual system of ER's corresponding to two incompatible observables is inferred from the *possibility* of measuring either of two

mutually incompatible observables†; this fact has been pointed out as 'the most significant lacuna' in EPR's reasoning [7]. For EPR's incompleteness argument to be valid [6, 7], *counterfactual definiteness* (CFD) [6, 9] is required, either as an additional assumption [7], or deduced from locality [9]. Independently of how natural it may appear, in our opinion this spoils EPR's own dictum: 'The elements of the physical reality cannot be determined by *a priori* philosophical considerations, but must be found by an appeal to results of experiments and measurements' [1].

   In their argument, EPR use the ambiguous expression 'can predict' in the condition for existence of ERs in a broad or 'weak' sense [6, 10–14], meaning that it is *possible* to make any of the *alternative* measurements that would provide the data for either inference. But, as it has been pointed out [6, 10–14], 'can predict' can also be interpreted in a narrow or 'strong' sense, meaning that we actually *do have* sufficient data to predict with certainty the *concrete* result to which the ER is ascribed. It is a common belief that EPR's argument does not work if this strong sense is required: 'The EPR argument goes through only if 'can predict' is understood in the weak sense' [14] (also see p 1885 of [10], or p 142 of [13]); EPR themselves said: 'Indeed, one would not arrive at our conclusion if one insisted that two or more physical quantities can be regarded as simultaneous elements of reality *only when they can be simultaneously measured or predicted*. On this point of view, since either one or the other, but not both simultaneously, of the quantities P and Q can be predicted, they are not simultaneously real. This makes the reality of P and Q depend upon the process of measurement carried out on the first system, which does not disturb the second system in any way. No reasonable definition of reality could be expected to permit this' [1]. The main aim of this paper is to show how we can formulate an EPR argument using ERs predicted in the strong sense‡.

3. Elements of reality inferred from joint measurements

We will call *strong elements of reality* (SERs) those obtained from EPR's criterion, taking 'can predict' in the strong sense. The name also reflects their *actuality*, in contradistinction with the ones used in EPR's paper and in most of the literature: SERs have definite predicted values, instead of an abstract existence without concrete values as the ERs inferred using the weak sense of 'can predict'. Note, however, that EPR's criterion is a *sufficient* condition, and that the set of assumptions used to infer SERs is in fact *weaker*: the CFD implicitly used in the original EPR argument is not required§.

† In [8], Redhead discusses how, if the *minimal instrumentalistic interpretation* of the QM formalism is complemented by any of several possible *views* (p 45), then 'the argument for incompleteness of the QM goes through without any consideration of the alternative possibilities of measuring [several incompatible observables]' (p 78). The locality principle and the inference of only one element of reality (in the singlet state, in Redhead's example) will be enough to prove QM incompleteness (p 77). In this paper we will prove how EPR's argument on QM incompleteness can be reformulated restricting ourselves to a minimal set of assumptions, both on QM interpretation (*minimal instrumentalistic interpretation*, in Redhead's terminology), and on the condition for existence of ERs.

‡ Such form of the argument will not work for systems with only two spacelike separated parts: in these systems, any couple of joint ERs predicted in the strong sense will correspond to one observable on each of the two spacelike separated parts, that are always compatible; in order to have ERs in the strong sense for incompatible observables we need three or more spacelike separated parts.

§ The word *counterfactual* is used in several contexts: quoting Ballentine [9], '*Counterfactual definiteness* (abbreviated CFD) . . . occurs in the EPR argument when they assert that if we had measured the position of particle 1 we could have learned the positions x_2 of particle 2, and if we had measured the momentum of particle 1 we could have learned the momentum p_2 of particle 2. Although only one of these measurements can

We will guarantee the clause 'without in any way disturbing a system' in EPR's sufficient condition by requiring that the system to which the ER is assigned be outside the future light cones of the points at which we perform any observation needed to make the prediction. This is how the ERs were inferred in EPR's original paper: 'since at the time of measurement the two systems no longer interact, no real change can take place in the second system in consequence of anything that may be done to the first system' [1]. More explicitly, we formulate the *sufficient* condition for existence of a SER in the following way.

Let us consider a physical system with two parts S_1, S_2, and two spacelike separated regions R_1, R_2 of the respective world tubes. If we can predict with certainty the concrete value of a physical quantity in R_2 from the result of a measurement performed in R_1, then there is a *strong element of reality* corresponding to that physical quantity, *at least* in the part of the world tube of S_2 outside the future light cones with vertices in R_1 and prior to any external perturbation of S_2.

The SER need not be the value of the physical quantity itself, as long as it determines univocally such value. For instance, in Bohm's theory the position of a particle in a Stern–Gerlach apparatus determines the value of the corresponding spin component [16]. Nevertheless, to simplify the discussion we will identify the SER with the value of the physical quantity that it predicts.

Our next step is to illustrate when several SERs can be jointly assigned to the same individual system. Suppose that we make a measurement in a region R_1 of the world tube of S_1, and that the result obtained allows us to infer a SER $r^{(2)}$ for S_2 following our previous criterion. On the same individual system, suppose that a second observer makes a measurement in a region R_2 of the world tube of S_2, spacelike separated from R_1, and that his result enables him to infer a SER $r^{(1)}$ for S_1 in its originally prepared state (prior to any disturbance). The persistence of the SER $r^{(2)}$ after the measurement at R_2 is not guaranteed (in general the measurement disturbs the subsystem S_2; it could even destroy it!); the same can be said about the SER $r^{(1)}$ after the measurement at R_1. According to the sufficient condition for existence of SERs, we can assign to that individual system two *joint* SERs, $r^{(1)}$ and $r^{(2)}$, at least in the parts of the world tubes outside the future light cones with vertices in R_1 and R_2 (loosely speaking, in the parts of the world tubes 'previous' to the measurements): a simplified case is shown in figure 1.

For any pair of (point-like) events, one in each region, there is always an inertial reference frame in which both events are simultaneous. Therefore, it is tempting to adopt the usual terminology and denote the joint SERs as 'simultaneous' SERs, but we have avoided this in order to emphasize that the combined existence of $r^{(1)}$ and $r^{(2)}$ in the same individual physical system does not depend on any external choice of a reference frame. We need not worry about instantaneity of measurements or simultaneity of three-dimensional sections of the four-dimensional world tubes of S_1 and S_2, and the generalization to systems with three or more extended spacelike separated parts is immediate: the only requisite on the compound system is that we can make predictions with certainty about properties (possibly non-local) of the individual system in some spacetime regions (prior to any external disturbance), based on measurements performed in other spacelike separated regions.

---

actually be carried out in a single case, the conclusion that both values x_2 and p_2 are well defined in nature is an instance of CFD.' *Counterfactual* is used also to qualify those predictions that are not actually tested (or even that *cannot* be tested); in this sense, we subscribe Pitowsky's opinion [15]: 'This [the fact that the experiment to test a prediction cannot be conducted] does not render the prediction invalid. It simply makes it untestable.' We think that counterfactually in this second sense is less worrying than the one involved in the first, where the *joint* inference of several ERs is based on *alternative* incompatible measurements.
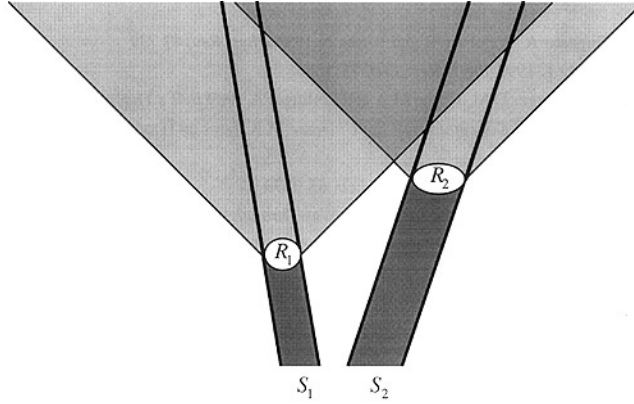
Figure 1. In order to simplify the representation we have assumed that the sections of the world tubes of subsystems S_1 and S_2 by the plane of the figure have straight well-defined borders and constant width (an obvious idealization, that disregards problems associated with non-compact support and spreading of the wavefunctions, or with more complicated geometries). A measurement made at the spacetime region, R_1, is used to infer an element of reality, $r^{(2)}$, for the subsystem, S_2, at least outside the future light cones with vertices in R_1, and prior to any external perturbation of S_2. Another measurement made at R_2 changes in general the state of subsystem S_2 (S_2 could even cease to exist!), and the persistence of $r^{(2)}$ as an element of reality is no longer guaranteed. The same reasoning applies when interchanging indices $1 \leftrightarrow 2$. Joint strong elements of reality $r^{(1)}$, $r^{(2)}$ exist at least in the parts of the world tubes outside the future light cones with vertices in R_1 and R_2 (darkest zones in the figure).

4. EPR's argument in terms of joint SER

We are going to follow EPR's previously quoted dictum on the necessity of finding ERs 'by an appeal to results of experiments', and reformulate their argument using only ERs jointly inferred from actual measurements (our previously defined joint SERs), without *a priori* use of alternative inferences. Joint SERs will be obtained for observables that have *no* common eigenstate; this is a stronger condition than the simple incompatibility; even if they do not have a basis of common eigenstates, two incompatible observables can share some eigenstates; the existence of joint SERs in one of these states will not prove the QM incompleteness (of course, $\hat{P}$, $\hat{Q}$ have no common eigenstate, but other incompatible observables without this property have been used in the literature).

Consider a system of three spin-$\frac{1}{2}$ particles in spacelike separated regions, prepared in the entangled spin state (in the basis of eigenstates of the operators $\hat{\sigma}_z^{(j)}$, $j = 1, 2, 3$),

$$|\Psi\rangle = a(|{+}{+}{+}\rangle - |{+}{-}{+}\rangle - |{-}{+}{+}\rangle) + b|{-}{-}{-}\rangle, \tag{1}$$

with $3|a|^2 + |b|^2 = 1$, and $ab \neq 0$.

In this state,

$$P_\Psi(\sigma_x^{(2)} = -1 | \sigma_z^{(1)} = +1) = 1, \tag{2}$$
$$P_\Psi(\sigma_x^{(1)} = -1 | \sigma_z^{(2)} = +1) = 1, \tag{3}$$

where $P_\Psi(\sigma_x^{(2)} = -1 | \sigma_z^{(1)} = +1)$ denotes the conditional probability of obtaining the value $\sigma_x^{(2)} = -1$ if $\sigma_z^{(1)} = +1$. If we measure $\hat{\sigma}_z^{(1)}$ on the first particle and obtain the result $+1$, property (2) allows us to assign a SER $\sigma_x^{(2)} = -1$ to the second particle, at least as long as it is not disturbed. Analogously, if a second observer measures $\hat{\sigma}_z^{(2)}$ on the second particle

and obtains the result $+1$, property (3) allows him to assign a SER $\sigma_x^{(1)} = -1$ to the first unperturbed particle†.

Consider now the non-local (and non-factorizable) observable defined by the projector

$$\hat{\pi}^{(1+2)} = 1 - |--\rangle\langle--|, \tag{4}$$

in the system formed by the first and second particles. This observable $\hat{\pi}^{(1+2)}$ is compatible with $\hat{\sigma}_z^{(1)}$ and $\hat{\sigma}_z^{(2)}$ but not with $\hat{\sigma}_x^{(1)}$ and $\hat{\sigma}_x^{(2)}$; in fact, it is easy to check that there is no common eigenstate to the three operators $\hat{\sigma}_x^{(1)} \otimes \mathbf{1}^{(2)}$, $\mathbf{1}^{(1)} \otimes \hat{\sigma}_x^{(2)}$ and $\hat{\pi}^{(1+2)}$.

In state (1),

$$P_\Psi(\pi^{(1+2)} = 1 | \sigma_z^{(3)} = +1) = 1. \tag{5}$$

Consequently, if a third observer measures $\hat{\sigma}_z^{(3)}$ and obtains the result $+1$, he can assign a SER $\pi^{(1+2)} = 1$ to the system formed by the first and second particles.

The probability of obtaining the results $\sigma_z^{(1)} = +1$, $\sigma_z^{(2)} = +1$, $\sigma_z^{(3)} = +1$ in a joint measurement of the three observables $\hat{\sigma}_z^{(j)}$ in spacelike separated regions of the same individual physical system is not zero:

$$P_\Psi(\sigma_z^{(1)} = +1, \sigma_z^{(2)} = +1, \sigma_z^{(3)} = +1) = |a|^2. \tag{6}$$

In an individual physical system in which these three results are obtained, following (2), (3) and (5) we can infer three joint SERs: $\sigma_x^{(1)} = -1$, $\sigma_x^{(2)} = -1$ and $\pi^{(1+2)} = 1$. Because there is *no* common eigenstate of the corresponding observables, according to EPR we would conclude that the quantum state 'does not provide a complete description of the physical reality' [1] of this individual system.

We could attempt a similar incompleteness argument in the state of the first two particles given by

$$|\eta\rangle = (|++\rangle - |+-\rangle - |-+\rangle)/\sqrt{3}, \tag{7}$$

which is an example of the Hardy state [2] that verifies properties (2) and (3) and is an eigenstate of $\hat{\pi}^{(1+2)}$ with eigenvalue 1. But then, the values $\sigma_x^{(1)} = -1$, $\sigma_x^{(2)} = -1$ and $\pi^{(1+2)} = 1$ would not satisfy our condition to be considered joint SERs, because the preparation of the state (7), in which the value $\pi^{(1+2)} = 1$ rest, is not spacelike separated from the measurements used to infer the two other values; the measurements of $\hat{\sigma}_z^{(1)}$, $\hat{\sigma}_z^{(2)}$ are in the future of the preparation of the state (7), and therefore their precise results in an individual system could be influenced by the preparation.

As we said in the second footnote on p 726, three or more spacelike separated parts are needed in order to meet the condition for existence of joint SERs for incompatible observables (of course, ERs might exist without this *sufficient* condition being fulfilled, but following EPR, we should not concern ourselves with more precise definitions of the ERs). The third particle in our example (1) is a device to allow us the use of the sufficient condition for existence of joint SERs, although the SERs inferred involve only the first two particles; in the next example the three particles play a more symmetrical role.

A stronger incompleteness proof can be worked out in the GMZ–Mermin [3,4] state of three spin-$\frac{1}{2}$ particles:

$$|\mu\rangle = (|+++\rangle - |---\rangle)/\sqrt{2}. \tag{8}$$

† Of course, after measuring $\sigma_z^{(1)}$, $\sigma_z^{(2)}$, the values predicted for $\sigma_x^{(1)}$, $\sigma_x^{(2)}$ cannot be verified, and in this sense these joint SERs are also counterfactual, but at least the measurements needed to infer both values can be made in the same individual system, and joint counterfactual (alternative) inferences are not involved; see third footnote on p 726.

Let us denote $\hat{A}_1 = \hat{\sigma}_x^{(2)} \otimes \hat{\sigma}_y^{(3)}$, $\hat{A}_2 = \hat{\sigma}_y^{(1)} \otimes \hat{\sigma}_x^{(3)}$, $\hat{A}_3 = \hat{\sigma}_x^{(1)} \otimes \hat{\sigma}_y^{(2)}$. In this state we have,

$$P_\mu(A_j = \varepsilon_j | \sigma_y^{(j)} = \varepsilon_j) = 1 \qquad \varepsilon_j = \pm 1, \ j = 1, 2, 3. \tag{9}$$

There is no common eigenstate to any two of the observables $\hat{A}_j$, and nevertheless we can infer joint SER for the three of them, by measuring $\hat{\sigma}_y^{(j)}$ in spacelike separated regions, whatever the results obtained in these measurements! (In our previous example we have to restrict ourselves to those individual physical systems in which $\sigma_z^{(j)} = +1$, $j = 1, 2, 3$.) The price to pay for this extension of the argument to all individual systems in state (8) is that the three incompatible observables to which the joint SER are assigned are now non-local (although factorizable), instead of one non-local and two local observables in the previous 'probabilistic' example.

In both examples (1), (8) the incompleteness argument has been formulated in terms of joint SERs, inferred from joint ('simultaneous', in the usual terminology) local measurements in causally non-connected regions of the same individual system, which was considered impossible until now [10, 13, 14].

5. The Bell–EPR theorem in terms of joint SER

In recent years there have been many proofs of the Bell–EPR theorem [17] 'without inequalities' [2–4, 11]. Arguments similar to those in the last section, in the same entangled states, lead us to proofs of these kind that are simple variants, using joint SERs, of Hardy's [2] and Mermin's [4] results.

State (1) has the following additional properties

$$P_\Psi(\sigma_z^{(2)} = -1 | \sigma_x^{(1)} = +1) = 1, \tag{10}$$

$$P_\Psi(\sigma_z^{(1)} = -1 | \sigma_x^{(2)} = +1) = 1, \tag{11}$$

$$P_\Psi(\sigma_x^{(1)} = +1, \sigma_x^{(2)} = +1, \sigma_z^{(3)} = +1) = \tfrac{1}{4}|a|^2. \tag{12}$$

If we measure the observables $\hat{\sigma}_x^{(1)}$, $\hat{\sigma}_x^{(2)}$ and $\hat{\sigma}_z^{(3)}$, and obtain $+1$ in the three cases (the probability for this is not zero, according to (12)), properties (10) and (11), together with property (5), allow us to infer, respectively, three joint SERs associated with the *compatible* observables $\hat{\sigma}_z^{(2)}$, $\hat{\sigma}_z^{(1)}$ and $\hat{\pi}^{(1+2)}$, with predicted values -1, -1 and 1, respectively. But according to QM, in a joint measurement (feasible in principle) of these observables in *any* state, if the first two results are $\sigma_z^{(1)} = \sigma_z^{(2)} = -1$, the third one will necessarily be $\pi^{(1+2)} = 0$: QM is not compatible with EPR's elements of reality, even in their less controversial form (joint SERs).

State (1) is the tensor product of the Hardy state (7) by state $|+\rangle$ of the third particle, entangled with the product of state $|--\rangle$ of the first two particles by state $|-\rangle$ of the third one. The presence of the third particle allows us to prove the incompatibility with QM (Bell's theorem) using three joint SERs that can never be found as results of the corresponding measurements, not only on state (1), but on *any* quantum state; by contrast, Hardy's two ERs, $\sigma_z^{(1)} = -1$, $\sigma_z^{(2)} = -1$, can never be obtained in a joint measurement in state (7), but states in which these results can be obtained do trivially exist.

The contradiction between ERs for compatible observables and QM found by Mermin [4] in the GHZ–Mermin state (8) can also be formulated in terms of joint SERs. Let us denote $\hat{B}_1 = \hat{\sigma}_y^{(2)} \otimes \hat{\sigma}_y^{(3)}$, $\hat{B}_2 = \hat{\sigma}_y^{(1)} \otimes \hat{\sigma}_y^{(3)}$, $\hat{B}_3 = \hat{\sigma}_y^{(1)} \otimes \hat{\sigma}_y^{(2)}$; then we have,

$$P_\mu(B_j = \varepsilon_j | \sigma_x^{(j)} = \varepsilon_j) = 1 \qquad \varepsilon_j = \pm 1, \quad j = 1, 2, 3, \tag{13}$$

$$P_\mu(\sigma_x^{(1)} \sigma_x^{(2)} \sigma_x^{(3)} = -1) = 1. \tag{14}$$

If the three $\hat{\sigma}_x^{(j)}$ are measured in spacelike separated regions of the same individual system in state (8), equation (13) allows us to infer three joint SERs, $B_j = \varepsilon_j$, that must verify the relation $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$ following equation (14). On the other hand, the product of the three *compatible* observables $\hat{B}_j$ is the unit operator, and therefore there is *no* quantum state in which the results of their measurements in the same individual system satisfy that relation (each result will be ± 1, but the product of the three is always $+1$). This proves again the incompatibility between QM and joint SERs.

Note that all observables used to infer ERs in section 4 and 5 are (local) spin components, and therefore there would be no difficulty for their joint measurement and the corresponding inference of joint SERs. Non-localities appear only in the observables to which the ERs are assigned; in this sense the experimental check of some of the predictions in sections 4 and 5 could be problematic, but our (gedanken) examples show the differences between QM *theory* and any theory that includes EPR's elements of reality: independently of any experimental confirmation of either theory, we have shown that ERs cannot be used to 'complete' QM.

6. Summary

Working only with what we have called 'joint SERs', inferred from joint measurements in the same individual physical system (a more *actual* kind than the usual ERs inferred from *alternative* incompatible measurements), we have reached the following conclusions.

(i) If we accept a very mild sufficient condition for the existence of elements of reality, joint SERs for observables without *any* common eigenstate will exist. On these premises, QM would be incomplete. This formulation of the EPR argument in terms of joint SERs was considered impossible.

(ii) There are sets of joint SERs for compatible observables that, according to QM, can never be obtained as results of the corresponding joint measurements in *any* individual system. However plausible, elements of reality are incompatible with QM (Bell's theorem).

Both theorems have been proved using quite similar arguments in the same *entangled* quantum states (1), (8).

Acknowledgments

References

[1] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777
[2] Hardy L 1993 *Phys. Rev. Lett.* **71** 1665
[3] Greenberger D M, Horne M A and Zeilinger A 1989 *Bell's Theorem, Quantum Theory, and Conceptions of the Universe* ed M Kafatos (Dordrecht: Kluwer) p 69
[4] Mermin N D 1990 *Phys. Rev. Lett.* **65** 3373
[5] Mermin N D 1990 *Boojums all the Way Through* (Cambridge: Cambridge University Press) p 177
[6] Hájek A and Bub J 1992 *Found. Phys.* **22** 313

[7]   Howard D 1985 *Stud. Hist. Phil. Sci.* **16** 171
[8]   Redhead M L G 1987 *Incompleteness, Nonlocality, and Realism* (Oxford: Clarendon)
[9]   Ballentine L E 1990 *Quantum Mechanics* (Englewood Cliffs, NJ: Prentice-Hall) p 453
[10] Clauser J F and Shimony A 1978 *Rep. Prog. Phys.* **41** 1881
[11] Greenberger D M, Horne M A, Shimony A and Zeilinger A 1990 *Am. J. Phys.* **5** 1131
[12] Stapp H P 1991 *Found. Phys.* **21** 1
[13] d'Espagnat B 1993 *Bell's Theorem and the Foundations of Modern Physics* ed A van der Merwe and F Selleri
         (Singapore: World Scientific) p 139
[14] Shimony A 1993 *Search for a Naturalistic World View* vol II (New York: Cambridge University Press) p 187
[15] Pitowsky I 1991 *Phys. Lett.* A **156** 137
[16] Bohm D and Hiley B J 1993 *The Undivided Universe* (London: Routledge)
[17] Bell J S 1964 *Physics* **1** 195